On an Open Problem of Amadio and Curien: the Finite Antichain Condition[1]

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Abstract

More than a dozen years ago, Amadio [1] (see Amadio and Curien [2] as well) raised the question of whether the category of stable bifinite domains of Amadio-Droste [1,6,7] is the largest cartesian closed full sub-category of the category of ω-algebraic meet-cpos with stable functions. A solution to this problem has two major steps: (1) Show that for any ω-algebraic meet-cpo D, if all higher-order stable function spaces built from D are ω-algebraic, then D is finitary (*i.e.*, it satisfies the so-called axiom I); (2) Show that for any ω-algebraic meet-cpo D, if D violates MI^{∞}, then $[D \to D]$ violates either M or I. We solve the first part of the problem in this paper, *i.e.*, for any ω-algebraic meet-cpo D, if the stable function space $[D \to D]$ satisfies M, then D is finitary. Our notion of (mub, meet)-closed set, which is introduced for step 1, will also be used for treating some example cases in step 2.

1 Introduction

The question of Amadio and Curien [1,2] arises in the delineation of the conceptual boundaries of stable domains. This alternative stable domains frame-

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work originates from Berry [3] who provides a domain-theoretic model for PCF with a better approximation to the notion of sequential computation. Berry's work was striking in that nobody at the time suspected that an alternative notion to Scott's continuous functions could have existed for a cartesian closed category of domains. Stable domains turned out to be of more general interest, playing significant roles in linear logic (Girard [11]), concurrency (Winskel [19]), type theory (Coquand [4]), and object-oriented programming (Reddy [16]).

The existence of a variety of cartesian closed categories of domains motivated a systematic investigation of the question of "largest cartesian closed categories of domains". The first step along this line of enquiry was made by Smyth [17], who showed that Plotkin's category SFP [15] is the largest cartesian closed one inside the category of ω-algebraic domains with Scott continuous functions. Substantial subsequent development followed in [13,14] (see [18] for most resent work). Similar development on stable domains has occurred only more recently. The first author showed in [22] that Berry's category of dI-domains is the largest cartesian closed category inside the category of Scott domains (which are bounded complete) with stable functions. In [1,5], appropriate notions of stable domains beyond the bounded complete ones were investigated, in an effort to provide an understanding of how the stable order may be extended to SFP-like domains. An interesting new category called *stable bifinite domains* was introduced. An important conceptual question (see Amadio and Curien [2], pages 287–291) is whether this category is the largest cartesian closed one within the category of ω-algebraic domains.

An affirmative answer to this question naturally breaks into two major steps (see Section 2 for the specifics of terminology):

(1) show that for any ω-algebraic meet-cpo D, if all higher-order stable function spaces built from D are ω-algebraic, then D is finitary (*i.e.*, it satisfies the so-called axiom I);
(2) show that for any ω-algebraic meet-cpo D satisfying axim I and axiom M, if D violates MI^∞ , then $[D \to D]$ violates either I or M. Here, axiom MI^∞ is an internal characterization of stable bifinite domains introduced by Amadio and Droste [1,5].

This paper introduces the method of $(\mathsf{mub}, \mathsf{meet})$-closed set as a way to solve the first part of the problem of Amadio and Curien: we show that for any ω-algebraic meet-cpo D, if the stable function space $[D \to D]$ satisfies M, then D is finitary. We also use the notion of $(\mathsf{mub}, \mathsf{meet})$-closed set to resolve some example cases for step 2.

Related work.

Amadio [1] and Droste [6,7] showed that the category of stable bifinite domains is cartesian closed. Amadio (Proposition 5.1 [1]) and Droste (Lemma 2.21 [7]) also showed that stable function space is the exponential object in any cartesian closed category using stable functions as morphisms. At a more technical level, Amadio [1] showed that for the stable function space $[D \to D]$ to be ω-algebraic, D must satisfy axiom M. He classified configurations violating axiom I into three sub-cases, and solved the cases of infinite ascending chain (below a compact element) and infinite descending chain, leaving open the infinite anti-chain (below a compact element) case. Proofs that identify M_∞ as a configuration for the infinite antichain case also appear in [12,22]. Fiore [9,10] demonstrates, from the axiomatic domain theory point of view, that there exist more liberal, cartesian closed categories based on an intensional notion of approximation. Stability can also be enriched this way, resulting in a cartesian closed category larger than stable bifinite domains. A similar question can be asked of a largest structure of this kind.

The rest of the paper is structured along the lines of background knowledge, $(\mathsf{mub}, \mathsf{meet})$-closed sets, the finite antichain condition for axiom I, and sample cases for axiom MI^∞.

2 Domains and stability

Leaving the rudimentary domain-theoretic definitions to [2], we begin with a description of *meet-cpos* as a general structure in which stability makes sense. We then recall the Amadio-Droste category **SB** of *stable bifinite domains*. By convention, we use $\downarrow x$ for the lower set $\{y \mid y \sqsubseteq x\}$ and $\uparrow x$ for the upper set $\{y \mid y \sqsupseteq x\}$. This notation extends to the upper set and lower set of sets, as $\uparrow A$ and $\downarrow A$, respectively. These are also called the *up-closure* and *down-closure* of A, respectively. A pair of elements $x, y \in D$ is called *bounded above*, or *compatible*, with $x \uparrow y$ in notation, if there exists an element $z \in D$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$. This notion easily extends to arbitrary sets instead of just a pair.

2.1 Meet-cpos

The basic property of a *conditional multiplicative* (cm) function is that it preserves the meet of any pair of compatible elements. Berry distinguishes cm functions from stable ones, those for which least local input information can be found for an achievable output. For the purpose of this paper, the notions of cm and stable functions are interchangeable, with conditional multiplicativity providing clean equational proofs, and minimal input providing intuition for stability. Sometimes we use the word cm simply to avoid a repetition of the word "stable".

Bounded meets should exist for stability to make sense (item (a) below). Meet should also interact smoothly with the join of any directed set (item (b) below). The stable order then arises naturally from the minimal requirement that the evaluation map (for cartesian closure) is stable [3].

Definition 2.1 *Let D be a cpo (with bottom). It is called a meet-cpo if*

(a) for any $x, y \in D$, $x \sqcap y$ exists when $\{x, y\}$ is bounded above (or compatible), and

(b) if $R \subseteq D$ is a directed set and x is compatible with the join of R, then

$$x \sqcap (\bigsqcup R) = \bigsqcup \{x \sqcap r \mid r \in R\}.$$

Note that the notion of meet-cpo is related to, but incompatible with the notion of meet-semi-lattice. In a meet-cpo we do not require meet to exist for all finite sets, but only for compatible pairs of elements. Further, when meet does exist, it should exhibit a continuity property as described by item (b) above. For algebraic domains, it suffices to require item (b) to hold for compact elements.

A simple but useful fact about meet-cpos will be used often in the rest of the paper.

Lemma 2.1 *Let D be a meet-cpo. If x, y are compatible minimal upper bounds of $\{a, b\}$, then $x = y$.*

Proof. Since x and y are compatible, $x \sqcap y$ exists. Since a is a lower bound of both x and y, we get $a \sqsubseteq x \sqcap y$ and, similarly, $b \sqsubseteq x \sqcap y$. This means $x \sqcap y$ is again an upper bound of $\{a, b\}$; and this can only happen when $x = y = x \sqcap y$ because x and y are both minimal upper bounds of $\{a, b\}$. □

Definition 2.2 *Let D, E be meet-cpos. A Scott continuous function $f : D \to E$ is called* stable *if it preserves meets of compatible pairs,* i.e., *for all x, y in D,*

$$x \uparrow y \Rightarrow f(x \sqcap y) = f(x) \sqcap f(y).$$

The stable function space $[D \to E]$ consists of all stable functions from D to E under the Berry order: f is stably less than g, written $f \sqsubseteq_s g$, if for all x, y in D,

$$x \sqsubseteq y \Rightarrow f(x) = f(y) \sqcap g(x).$$

Notation. We exclusively deal with the stable function space (*i.e.,* cm functions under the stable order) in this paper and will use $[D \to E]$ as the default notation for it.

The next result can be found in [1].

Theorem 2.1 *The category of meet-cpos with stable functions is a cartesian closed category (ccc).*

We mention below some basic properties of stable functions, the proofs of which can be found in either [3] or [24].

Lemma 2.2 *Let D, E be meet-cpos and f, g be compatible stable functions in $[D \to E]$. We have*

(a) if $f(x) = g(x)$, then $f(y) = g(y)$ for any $y \in {\downarrow}x$,
(b) if $a \uparrow b$ then $f(a) \sqcap g(b) = f(b) \sqcap g(a)$.

This lemma reveals a striking difference between the stable order and the standard extensional order: if compatible stable functions share the same value at some point, they must be identical on the principle ideal determined by that point. The contrapositive of this observation is also worth noting:

> *If $f(x) = g(x)$ but $f(y) \neq g(y)$ for some $y \sqsubseteq x$, then f and g are incompatible with respect to the stable order.*

Here is also the natural place for the next lemma, which will be needed in Section 4. It states that meet can be computed point-wise for bounded stable function pairs, and hence for finite sets of bounded stable functions.

Lemma 2.3 *Let D, E be meet-cpos. If stable functions $f, g : D \to E$ are bounded above, then their meet is determined point-wise, i.e.,*

$$f \sqcap g = \lambda x. f(x) \sqcap g(x).$$

The proof is tedious but routine, hence omitted. It is similar in spirit to Lemma 6.6 of Zhang [24] and makes use of Proposition 8.9 of Winskel [20].

Note that a similar statement for join ($\sqcup$) does not hold in general. Any two functions h_i, h_j given in Lemma 4.5 are bounded above. But the $g_{n,m}$'s are all *distinct minimal upper bounds*, instead of *the* least upper bound. However, directed joins can be achieved point-wise for stable functions, as stated in the next lemma.

Lemma 2.4 *Let D, E be meet-cpos. Suppose $\{f_i \mid i \in I\}$ is a directed family of stable functions in $[D \to E]$. Then its join is determined point-wise, i.e., $(\bigsqcup_{i\in I} f_i)(x) = \bigsqcup_{i\in I} f_i(x)$ for all $x \in D$.*

The proof is similar to one given in [24] for dI-domains. But since this property is used later on, we explain the key steps of the proof. Note that by similar techniques in [20], the function determined by point-wise join is already Scott continuous. The point-wise join is also stable since D and E are meet-cpos. It suffices to check that it is indeed an upper bound in the stable order. Let $x \sqsubseteq y$ in D. We have, for any $k \in I$,

$$\begin{aligned} & f_k(y) \sqcap \textstyle\bigsqcup_{i\in I} f_i(x) \\ = & \textstyle\bigsqcup_{i\in I}(f_k(y) \sqcap f_i(x)) \quad \textit{(D a meet-cpo)} \\ = & \textstyle\bigsqcup_{i\in I}(f_k(x) \sqcap f_i(y)) \quad \textit{(Lemma 2.2, (b) and directedness)} \\ \sqsubseteq & f_k(x). \end{aligned}$$

Therefore,

$$f_k(x) = f_k(y) \sqcap \bigsqcup_{i\in I} f_i(x).$$

2.2 Stable bifinite domains

Meet-cpos need not be ω-algebraic. Amadio [1] and Droste [6,7] showed that beyond Scott domains, there is the category of stable bifinite domains which also forms a ccc. However, readers should be aware of the small notational variations of the terminology *stable bifinite domains* in the literature. For example, in [2], Section 12.4 refers to stable bifinite domains without requiring a countable basis (i.e., ω-algebracity), which results in a cartesian closed category (c.f. Prop. 12.4.4 [2]) while Droste [6] showed the ccc result for stable ω-bifinite L-domains. With respect to the quest for a maximal ccc of stable domains in this paper, we consider ω-algebraic domains as the ambient category, and the countability of base elements is a prerequisite.

By stable bifinite domains we mean ω-algebraic meet-cpos for which the identity function can be expressed as the join (under the stable order) of a directed set of stable projections with finite images. For the purpose of this paper, we take an internal characterization of stable bifinite domains as the definition.

Definition 2.3 (Property M) *Let D be an ω-algebraic meet-cpo and X a subset of D. The set of minimal upper bounds (mubs) of X is denoted as $\mathsf{mub}(X)$. D is said to have property M if for every finite set X of compact elements, $\mathsf{mub}(X)$ is finite and complete – complete in the sense that each* *upper bound of X dominates some members of* $\mathsf{mub}(X)$*. Property* M *is also called the "2/3 SFP" condition.*

Let $\bowtie(X) := \bigcup\{\mathsf{mub}(Y) \mid Y \subseteq_{\text{fin}} X\}$, where $Y \subseteq_{\text{fin}} X$ means that Y is a finite subset of X. A set is called *mub-closed* if $\bowtie(X) = X$. An SFP domain, according to Plotkin [15], is an ω-algebraic cpo with property M, such that every finite set X of compact elements is contained in a finite mub-closed set. Stable bifinite domains are similar to SFP domains, but a stronger condition holds: for any finite set of compact elements, there is a finite superset, closed under the combination of down-closure and mub-closure. More precisely, let $(\mathsf{mub}, \mathsf{down})(X) := \downarrow(\bowtie(X))$. A set X is called *mub-down-closed* if $(\mathsf{mub}, \mathsf{down})(X) = X$.

Definition 2.4 (Stable Bifinite Domain) *An ω-algebraic meet-cpo is said to have property* I *and called finitary if every compact element dominates a finite number of elements. It is called a* **stable bifinite domain** *if every finite set of compact elements is contained in a finite* $(\mathsf{mub}, \mathsf{down})$*-closed set. This last property is denoted as* MI^{∞} *.*

Let **SB** be the category of stable bifinite domains with stable functions (under the Berry order for function space). We have the following [1,5].

Theorem 2.2 *The category* **SB** *is a cartesian closed category.*

Additional useful properties given in [1] include the following.

Lemma 2.5 *Let D be a finitary ω-algebraic meet-cpo. Then for any finite set X of compact elements, we have $\bowtie(\bowtie(X)) = \bowtie(X)$.*

Lemma 2.5 implies that, since both $\bowtie$ and $\downarrow(\ )$ are closures, the alternation between $\bowtie$ and $\downarrow(\ )$ is necessary to account for the $(\mathsf{mub}, \mathsf{down})$-closure.

Lemma 2.6 *If both D and $[D \to D]$ are ω-algebraic meet-cpos, then D satisfies* M*.*

Based on this lemma, we can safely assume that all domains are ω-algebraic meet-cpos with property M for the rest of the paper.

3 (mub, meet)-closed sets

We now introduce the main technical notion of the paper: $(\mathsf{mub}, \mathsf{meet})$-closed sets.

Definition 3.1 *Suppose D is an ω-algebraic meet-cpo with property M. A set Y of compact elements of D is said to be a $(\mathsf{mub}, \mathsf{meet})$-closed set if both of the following are true:*

(a) it is closed under minimal upper bounds of finite sets,

(b) it is closed under meets of compatible pairs of elements.

Item (a) deliberately allows the inclusion of the mub of the empty set, which is the bottom of D. Thus every $(\mathsf{mub}, \mathsf{meet})$-closed set contains the bottom. Item (b) says that for $x, y \in Y$ with $x \uparrow y$, we have $x \sqcap y \in Y$. Let $\mathsf{meet}(X) := \{x \sqcap y \mid x, y \in X \ \& \ x \uparrow y\}$. The operator $(\mathsf{mub}, \mathsf{meet})$ can be defined as $(\mathsf{mub}, \mathsf{meet})(X) := \mathsf{meet}(\bowtie(X))$.

Clearly, every $(\mathsf{mub}, \mathsf{down})$-closed set is $(\mathsf{mub}, \mathsf{meet})$-closed. Moreover,

$$(\mathsf{mub}, \mathsf{meet})(X) \subseteq (\mathsf{mub}, \mathsf{down})(X)$$

for every X. However, $(\mathsf{mub}, \mathsf{meet})$-closed sets provide a more flexible and general way for constructing stable functions.

Lemma 3.1 *Suppose D is an ω-algebraic meet-cpo with property M. Then every $(\mathsf{mub}, \mathsf{meet})$-closed set A determines a stable function $\phi_A : D \to D$, given by*

$$\phi_A(x) := \bigsqcup(\downarrow x \cap A)$$

for each $x \in D$.

Proof. Property M and mub-closedness of A ensure that $\downarrow x \cap A$ is a directed set for any $x \in D$; hence ϕ_A is well-defined. The continuity of ϕ_A follows from the assumption that A consists of compact elements. We check stability in detail. Let $x \uparrow y$ in D. We need to show that

$$(\bigsqcup \downarrow x \cap A) \sqcap (\bigsqcup \downarrow y \cap A) = \bigsqcup \downarrow(x \sqcap y) \cap A.$$

This is true because by the continuity of meet in a meet-cpo, we have

$$
\begin{aligned}
& (\bigsqcup \downarrow x \cap A) \sqcap (\bigsqcup \downarrow y \cap A) \\
=\ & \bigsqcup \{ (\bigsqcup \downarrow x \cap A) \sqcap b \ \mid \ b \in \downarrow y \cap A \} \\
=\ & \bigsqcup \{ a \sqcap b \mid a \in \downarrow x \cap A,\ b \in \downarrow y \cap A \} \\
=\ & \bigsqcup \{ c \mid c \in \downarrow (x \sqcap y) \cap A \}
\end{aligned}
$$

where the last step follows from the assumption that A is closed under compatible meet. □

An immediate consequence of this lemma is that $(\mathsf{mub}, \mathsf{down})$-closed sets determine stable functions. These are projections, which are idempotent under functional composition and dominated by the identity function in the stable order. *However,* $(\mathsf{mub}, \mathsf{meet})$*-closed sets do not determine projections in general; they are idempotent under functional composition but may not be subsumed by the identity function in the stable order.*

Lemma 3.2 *Let ϕ_A be the stable function determined by a* $(\mathsf{mub}, \mathsf{meet})$*-closed set A as given in the previous lemma. Then*

(a) A is the set of compact fixed-points of ϕ_A, and
(b) if $f \sqsubseteq \phi_A$ and $f(x) = x$ for each $x \in A$, then $f = \phi_A$.

Proof. Suppose x is a compact fixed-point of ϕ_A. Then $x = \bigsqcup(\downarrow x \cap A)$. Since x is compact and $\downarrow x \cap A$ is directed, $x \sqsubseteq y$ for some $y \in \downarrow x \cap A$. This is only possible if $x = y$, which implies $x \in A$ in turn. This proves (a).

For (b), suppose $f \sqsubseteq \phi_A$ and $f(x) = x$ for each $x \in A$. It suffices to show that $\phi_A(y) \sqsubseteq f(y)$ for each $y \in D$. This is true because

$$
\begin{aligned}
& \phi_A(y) & \\
=\ & \bigsqcup \{ x \mid x \in A \ \&\ x \sqsubseteq y \} & \text{(definition)} \\
=\ & \bigsqcup \{ f(x) \mid x \in A \ \&\ x \sqsubseteq y \} & (f(x) = x \text{ for } x \in A) \\
\sqsubseteq\ & \bigsqcup \{ f(y) \mid x \in A \ \&\ x \sqsubseteq y \} & (f(x) \sqsubseteq f(y)) \\
=\ & f(y) &
\end{aligned}
$$

□

The next lemma shows how stable functions determined by $(\mathsf{mub}, \mathsf{meet})$-closed sets can be compared.

Lemma 3.3 *Suppose A, B are $(\mathsf{mub}, \mathsf{meet})$-closed sets of an ω-algebraic meet-cpo with property M. Then the following are equivalent:*

(a) $\phi_B \sqsubseteq_s \phi_A$,

(b) $\downarrow B \cap A = B$,

(c) $B \subseteq A$ *and for each compatible pair of elements* x, y, *if* $x \in B$ *and* $y \in A$, *then* $x \sqcap y \in B$.

Proof. $(a) \Rightarrow (b)$: Suppose A, B are $(\mathsf{mub}, \mathsf{meet})$-closed sets with $\phi_B \sqsubseteq_s \phi_A$. Let y be a member of $\downarrow B \cap A$. Then $y \in A$ and $y \sqsubseteq x$ for some $x \in B$. Since $\phi_B(y) = \phi_B(x) \sqcap \phi_A(y)$, we have $\phi_B(y) = x \sqcap y = y$. By Lemma 3.2, $y \in B$. Thus $\downarrow B \cap A \subseteq B$. On the other hand, let $y \in B$. Then $y = \phi_B(y) \sqsubseteq \phi_A(y) \sqsubseteq y$. Therefore, $y \in A$ by Lemma 3.2. This shows $B \subseteq \downarrow B \cap A$.

$(b) \Rightarrow (c)$: Suppose $\downarrow B \cap A = B$. Then clearly $B \subseteq A$. Suppose x, y are compatible, with $x \in B$ and $y \in A$. Then $x \sqcap y$ remains a member of A since A is $(\mathsf{mub}, \mathsf{meet})$-closed. Moreover, $x \sqcap y \in \downarrow B$ since $x \in B$. Therefore, $x \sqcap y$ is a member of $\downarrow B \cap A$. By the assumption that $\downarrow B \cap A \subseteq B$, we have $x \sqcap y \in B$, as required.

$(c) \Rightarrow (a)$: Assume that $B \subseteq A$ and for each compatible pair of elements x, y, if $x \in B$ and $y \in A$, then $x \sqcap y \in B$. We immediately have $\phi_B \sqsubseteq \phi_A$ by the definition in Lemma 3.1 because $B \subseteq A$. To show the stable order, $\phi_B \sqsubseteq_s \phi_A$, let $a \sqsubseteq b$ in D and we need to show that $\phi_B(b) \sqcap \phi_A(a) \sqsubseteq \phi_B(a)$. This is true because

$$\begin{aligned}
&\phi_B(b) \sqcap \phi_A(a) \\
&= \bigsqcup\{x \mid x \in B \;\&\; x \sqsubseteq b\} \sqcap \bigsqcup\{y \mid y \in A \;\&\; y \sqsubseteq a\} \\
&= \bigsqcup\{x \sqcap y \mid x \in B \;\&\; x \sqsubseteq b \;\&\; y \in A \;\&\; y \sqsubseteq a\} \qquad \text{(use Def. 2.1 twice)} \\
&\sqsubseteq \bigsqcup\{z \mid z \in B \;\&\; z \sqsubseteq a\} \\
&= \phi_B(a)
\end{aligned}$$

where the last step follows from the fact that if $x \in B, x \sqsubseteq b, y \in A$ and $y \sqsubseteq a$, then $x \sqcap y \in B$ (by the given assumption) and $x \sqcap y \sqsubseteq a$. □

When a set X of compact elements is not already $(\mathsf{mub}, \mathsf{meet})$-closed, we can work with the $(\mathsf{mub}, \mathsf{meet})$-closed set *generated* by X, which is the smallest set of compact elements containing X and closed under minimal upper bounds of finite subsets and finite bounded meets. Such a generated set always exists in an ω-algebraic meet-cpo with both property M and the property that the meet of two compact elements is compact. In such a case the closure exists and can be defined inductively as:

$$\begin{aligned}
(\mathsf{mub},\mathsf{meet})^0(X) &:= X\\
(\mathsf{mub},\mathsf{meet})^{(i+1)}(X) &:= (\mathsf{mub},\mathsf{meet})((\mathsf{mub},\mathsf{meet})^i(X))\\
(\mathsf{mub},\mathsf{meet})^*(X) &:= \bigcup_{i\geq 0}(\mathsf{mub},\mathsf{meet})^i(X)
\end{aligned}$$

Clearly, the set $(\mathsf{mub},\mathsf{meet})^*(X)$ so obtained is the least $(\mathsf{mub},\mathsf{meet})$-closed set containing X.

Lemma 3.4 *Suppose the meet of a compatible pair of compact elements remains compact in D. Let Y be the $(\mathsf{mub},\mathsf{meet})$-closed set generated by a finite set Y^0 of compact elements of D. Then the stable function ϕ_Y is compact in $[D \to D]$.*

Proof. Suppose $\phi_Y \sqsubseteq_s \bigsqcup_{i\in I} f_i$, where $\{f_i \mid i \in I\}$ is a directed set of stable functions. By Theorem 2.1, the stable function space is again a meet cpo. Continuity of binary meet in the function space allows us to obtain that

$$\phi_Y \sqcap \phi_Y \sqsubseteq_s \phi_Y \sqcap (\bigsqcup_{i\in I} f_i) = \bigsqcup_{i\in I}(\phi_Y \sqcap f_i) \sqsubseteq_s \phi_Y.$$

Thus we may assume (by replacing f_i by $\phi_Y \sqcap f_i$) that $\phi_Y = \bigsqcup_{i\in I} f_i$ with $f_i \sqsubseteq_s \phi_Y$ for all $i \in I$ and show that $\phi_Y = f_k$ for some $k \in I$.

We achieve this by induction on the number of iterations of the $(\mathsf{mub},\mathsf{meet})$-operator on Y^0, that there exists $k \in I$ such that for all $i \geq 0$, $\phi_Y(b) = f_k(b)$ for all $b \in (\mathsf{mub},\mathsf{meet})^i(Y^0)$. Lemma 3.2 then implies that $\phi_Y = f_k$.

Basis. For each $b \in Y^0$, since $b = \phi_Y(b) = \bigsqcup_{i\in I} f_i(b)$ (by Lemma 2.4), and b is compact, there exists an $i(b)$ such that $b = \phi_Y(b) = f_{i(b)}(b)$. There are finitely many elements in Y^0; this allows us to choose k to be an upper bound of the set $\{i(b) \mid b \in Y^0\}$ so that $\phi_Y(b) = f_k(b)$ for all $b \in Y^0$, as required for the basis step.

Inductive step. Assume $\phi_Y(b) = f_k(b)$ for all $b \in (\mathsf{mub},\mathsf{meet})^i(Y^0)$ for some $i \geq 0$. Let m be a minimal upper bound of a subset X of $(\mathsf{mub},\mathsf{meet})^i(Y^0)$. Since $f_k(b) = \phi_Y(b) = b$ for all $b \in X$, $f_k(m) \sqsupseteq f_k(b)(= b)$ for all $b \in X$ since f_k is monotone. Thus $f_k(m)$ is an upper bound of X. On the other hand, $m \in Y$ and hence $\phi_Y(m) = m$ by Lemma 3.2. This means $f_k(m) \sqsubseteq \phi_Y(m) = m$, forcing $f_k(m) = m$ since m is a mub of X. This concludes that ϕ_Y and f_k agree on all the mubs of subsets of $(\mathsf{mub},\mathsf{meet})^i(Y^0)$. Once this is achieved, ϕ_Y and f_k are forced to agree on the down-closure of these (new) mubs by the remarks following Lemma 2.2, noting the compatibility of ϕ_Y and f_k. This shows that $\phi_Y(b) = f_k(b)$ for all $b \in (\mathsf{mub}, \mathsf{meet})^{i+1}(Y^0)$, as needed for the inductive step. $\square$

A non-trivial case is the $(\mathsf{mub}, \mathsf{meet})$-closure of a two-element set, which remains finite (Fig. 1) in an ω-algebraic meet-cpo with property M.

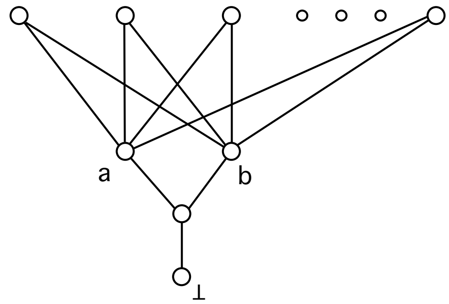

Fig. 1. $(\mathsf{mub}, \mathsf{meet})$-closure of two elements a, b.

Lemma 3.5 *Let D be an ω-algebraic meet-cpo with property* M *and the property that the meet of a compatible pair of compact elements remains compact. Then every pair of compact elements generates a finite* $(\mathsf{mub}, \mathsf{meet})$*-closed set.*

Proof. This is because for two compatible compact elements a, b, the set $\{\bot, a \sqcap b, a, b\} \cup (\mathsf{mub}\{a, b\})$ is finite and $(\mathsf{mub}, \mathsf{meet})$-closed. (Note that distinct elements in $\mathsf{mub}\{a, b\}$ are incompatible in a meet cpo.) $\square$

We conclude this section with the example below, showing that the $(\mathsf{mub}, \mathsf{meet})$-closure of a three-element set can be infinite. The starting three elements are marked solid black (see Fig 2).

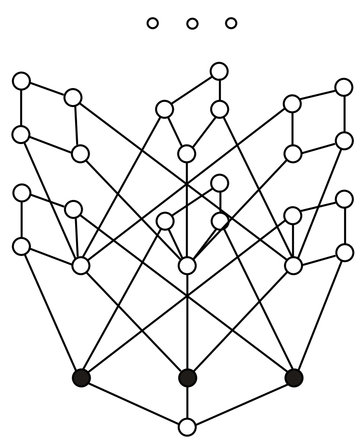

Fig. 2. Infinity of $(\mathsf{mub}, \mathsf{meet})$-closure

4 The finite antichain condition

The main result of this section is that for a full sub-category of ω-algebraic meet-cpos to be cartesian closed, all of its domains must be finitary (Theorem 4.1).

With respect to an ω-algebraic meet-cpo D, property I breaks down to three more primitive cases (see [1], where these cases are called I_1, I_2, and I_3, respectively, corresponding to (a), (b), (c) below, with (c) stated differently) with respect to principle ideals generated by a compact element:

(a) *the finite descending chain condition*: for any compact element $a \in D$, the principle ideal $\downarrow a$ does not contain an infinite descending chain of compact elements,
(b) *the finite ascending chain condition*: for any compact element $a \in D$, the principle ideal $\downarrow a$ does not contain an infinite ascending chain of compact elements, and
(c) *the finite antichain condition*: for any compact element $a \in D$, the principle ideal $\downarrow a$ does not contain an infinite antichain of compact elements.

Amadio [1] showed that if an ω-algebraic meet-cpo fails property I, then it fails either the finite descending chain condition, or the finite ascending chain condition, or the finite antichain condition. He also resolved the first two subcases in [1] (see [22] as well).

Before getting to the main result of this section, note that the method of $(\mathsf{mub}, \mathsf{meet})$-closed set provides a crisp construction of an uncountable basis in the stable function space $[D \to D]$ when D fails either the finite descending chain condition or the finite ascending chain condition. We provide detailed proof for Lemma 4.1 to illustrated the value of $(\mathsf{mub}, \mathsf{meet})$-closed sets.

Lemma 4.1 *Suppose D is an ω-algebraic meet-cpo with property M. If D fails the finite descending chain condition, then $[D \to D]$ is not ω-algebraic.*

Proof. Suppose, for some compact element a_0, the principle ideal $\downarrow a_0$ contains an infinite descending chain of compact elements $a_1 \sqsupset a_2 \sqsupset \ldots \sqsupset a_i \ldots$ It is clear that for any subset $A \subseteq \{a_i \mid i \geq 1\}$, the set $B := A \cup \{\perp, a_0\}$ is a $(\mathsf{mub}, \mathsf{meet})$-closed set. Lemma 3.1 tells us that each ϕ_B is a stable function. It suffices to show that each distinct B determines a distinct compact stable function, resulting in an uncountable number of compact stable functions in $[D \to D]$.

That distinct B's determine distinct (in fact incompatible) stable functions follows from the remarks after Lemma 2.2, since all the ϕ_B's agree on a_0.

We show that ϕ_B is compact. As in the proof of Lemma 3.4, let us assume, without loss of generality, that $\phi_B = \bigsqcup_{i\in I} f_i$ for a directed family of stable functions $\{f_i \mid i \in I\}$ in $[D \to D]$, where $f_i \sqsubseteq \phi_B$ for all $i \in I$. We have $a_0 = \phi_B(a_0) = \bigsqcup_{i\in I} f_i(a_0)$, by Lemma 2.4. Since a_0 is compact, $a_0 \sqsubseteq f_k(a_0) (\sqsubseteq \phi_B(a_0))$ for some $k \in I$. Thus $f_k(a_0) = a_0$.

Now, for any $b \in B$, we have $f_k(b) = f_k(a_0) \sqcap \phi_B(b) = b$ because $f_k \sqsubseteq_s \phi_B$ and $b \sqsubseteq a_0$. We then apply Lemma 3.2 to obtain $f_k = \phi_B$. □

With a similar proof, one can show the next lemma.

Lemma 4.2 *Suppose D is an ω-algebraic meet-cpo with property* M. *If D fails the finite ascending chain condition, then $[D \to D]$ is not ω-algebraic.*

The most difficult case arises when D satisfies the finite descending chain condition and the finite ascending chain condition, but fails the finite antichain condition. This open problem has resisted a resolution for more than a decade, and we finally are able to solve it here after a number of failed (in subtle ways) initial attempts. Figure 3 is a picture of a configuration violating this condition.

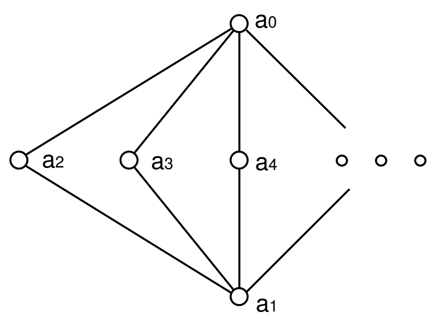

Fig. 3. M_∞

As stated in the next lemma, such a structure is present in any ω-algebraic meet-cpo which satisfies both the finite descending chain and the finite ascending chain conditions but not the finite antichain condition. Similar lemmas can be found in [1,12,22]. We provide a new (but non-constructive) proof which allows us to achieve the stronger conclusion that the a_i for $i \geq 2$ are all *covers* of a_1, i.e., there are no elements strictly between a_i and a_1.

Lemma 4.3 *Suppose D is an ω-algebraic meet-cpo with property* M *which satisfies both the finite descending chain and the finite ascending chain conditions, but fails axiom* I. *Then there exist $a_0, a_1 \in D^0$ and an infinite antichain of compact elements $\{a_i \mid i \geq 2\}$ such that*

(a) a_i covers a_1 for each $i \geq 2$,

(b) $a_i \sqcup^{a_0} a_j = a_0$ *for each* $i \neq j$ *and* $i, j \geq 2$*, and*
(c) $\downarrow a_i$ *is finite for each* $i \geq 2$.

Here $x \sqcup^a y$ *stands for the least upper bound of* x *and* y *in the principal ideal* $\downarrow a$*, which exists for any compact elements* x, y *in the principal ideal* $\downarrow a$.

Proof. Under the given conditions, let a_0 be a minimal compact element such that $\downarrow a_0$ is infinite. There must be infinitely many elements covered by a_0, for otherwise a_0 would not be the minimal compact element such that $\downarrow a_0$ is infinite.

Inside $\downarrow a_0$, assume a_1 to be a maximal element such that infinitely many elements in $\downarrow a_0$ *cover* a_1. Such an element always exists by König's Lemma and the given assumptions.

Now let $\{b_i \mid i \geq 0\}$ be the infinite set of elements in $\downarrow a_0$ such that b_i covers a_1 for each $i \geq 0$. This satisfies (a) and (c). We need to find an infinite subset of $\{b_i \mid i \geq 0\}$ that satisfies (b). Let $\{a_i \mid i \in I\}$ be a maximal subset of $\{b_i \mid i \geq 0\}$ with property (b). By Zorn's Lemma, such a maximal subset exists. It suffices to show that $\{a_i \mid i \in I\}$ is infinite. Suppose $\{a_i \mid i \in I\}$ is a finite, maximal subset of $\{b_i \mid i \geq 0\}$ with property (b). Consider the set of sups $\{a_i \sqcup^{a_0} b_j \mid i \in I \ \&\ j \notin I\}$. Since $\{a_i \mid i \in I\}$ is maximal, for each $j \notin I$, there exists $i \in I$, such that $a_i \sqsubset a_i \sqcup^{a_0} b_j \sqsubset a_0$ (note that the $a_i \sqsubset a_i \sqcup^{a_0} b_j$ part is always true because both a_i and b_j cover a_1). By the maximality of a_1, the set $(\downarrow\{a_0\}) \cap \uparrow\{a_i \mid i \in I\}$ is finite. Therefore, the mub set

$$\{a_i \sqcup^{a_0} b_j \mid i \in I \ \&\ j \notin I \ \&\ a_i \sqcup^{a_0} b_j \sqsubset a_0\}$$

is finite. Hence, for some $k \in I$, $a_k \sqcup^{a_0} b_m = a_k \sqcup^{a_0} b_n (:= a)$ for infinitely many distinct $m, n \notin I$. Thus $b_i \sqsubseteq a$ for infinitely many b_i's. This contradicts the minimality assumption of a_0 stated at the beginning of the proof. □

For the rest of the section, we assume that D is an ω-algebraic meet-cpo which satisfies both the finite descending chain condition and the finite ascending chain condition, but fails I. We also assume that D contains an infinite antichain $\{a_i \mid i \geq 0\}$ of compact elements as described in Lemma 4.3.

Lemma 3.1 and Lemma 3.5 motivate our next lemma, critical for Theorem 4.1.

Lemma 4.4 *Suppose the set* $\{a_i \mid i \geq 0\}$ *is as given in Lemma 4.3. For* $i, j \geq 2$*, let* $g_{i,j} := \phi_{A_{i,j}}$*, where* $A_{i,j}$ *is the finite* $(\mathsf{mub}, \mathsf{meet})$*-closed set generated by* $\{a_i, a_j\}$ *(by Lemma 3.5), consisting of* $\perp$*,* a_1*,* a_i*,* a_j*, and minimal upper bounds of* $\{a_i, a_j\}$*. We have*

(a) $g_{i,j}$ *is a compact stable function, and*
(b) $g_{i,j} \uparrow g_{s,t}$ *if and only if* $\{i,j\} = \{s,t\}$*.*

Proof. (a) follows from Lemma 3.1 and Lemma 3.4. We prove (b). Suppose $\{i,j\} \neq \{s,t\}$, and, suppose $i \notin \{s,t\}$, without loss of generality. We have $g_{i,j}(a_0) = g_{s,t}(a_0) = a_0$ by definition, but $g_{i,j}(a_i) = a_i \neq g_{s,t}(a_i) = a_1$. By item (a) of Lemma 2.2, we have $g_{i,j} \not\uparrow g_{s,t}$. □

Our earlier failed attempts relied too much on functions given by $(\mathsf{mub}, \mathsf{meet})$-closed sets as described in Lemma 3.1. Doing so entailed focusing on the mubs of functions generated by sets of the form $\{a_1, a_i\}$, $i \geq 2$ which led to nowhere. A slightly modified version seems to be the key, as given in the next lemma, but an explicit definition of $g_{i,j}$ should be helpful at this point:

$$g_{i,j}(x) := \begin{cases} a_i, & \text{if } x \in (\uparrow a_i - \uparrow a_j) \\ a_j, & \text{if } x \in (\uparrow a_j - \uparrow a_i) \\ b, & \text{if } x \in (\uparrow a_i \cap \uparrow a_j) \\ & \quad \text{and } x \sqsupseteq b \in \mathsf{mub}(\{a_i, a_j\}) \\ a_1, & \text{if } x \in \uparrow a_1 - (\uparrow a_i \cup \uparrow a_j) \\ \bot & \text{if } x \notin \uparrow a_1 \end{cases}$$

Lemma 4.5 *Suppose the set* $\{a_i \mid i \geq 0\}$ *is as given in Lemma 4.3. For* $k \geq 2$*, define functions* $h_k : D \to D$ *by*

$$h_k(x) := \begin{cases} a_k, & \textit{if } x \in \uparrow a_0 & (\alpha) \\ a_1, & \textit{if } x \in (\uparrow a_1 - \uparrow a_0) & (\beta) \\ \bot, & \textit{if } x \notin \uparrow a_1 & (\gamma) \end{cases}$$

Then the following are true:

(a) *each* h_k *is a compact stable function,*
(b) $g_{i,j}$ *is a minimal upper bound of* $\{h_k, h_l\}$ *in the stable order for all distinct* $i, j, k, l \geq 2$*.*

Proof. It is easy to see that each h_k is a compact stable function. So we focus on (b). To show that $g_{i,j}$ is an upper bound of $\{h_k, h_l\}$, note that extensionally, we have $h_m(x) \sqsubseteq g_{i,j}(x)$ for all $x \in D$ and all $m \geq 2$. To check the condition for the stable ordering, let $x \sqsubseteq y$ in D. Since $g_{i,j}(z) = h_m(z)$ for all $z \notin \uparrow a_1$, the only non-trivial case to check is when x comes from region β (i.e., $\uparrow a_1 - \uparrow a_0$) and y comes from region α (i.e., $\uparrow a_0$) in the definition of h_m. In this case $h_m(x) = a_1$, $h_m(y) = a_m$, while the value of $g_{i,j}(x)$ depends on the location of x in region β (i.e., $\uparrow a_1 - \uparrow a_0$). We need to check that under the given conditions, the possible values for $g_{i,j}(x)$ are a_i, a_j, and a_1, and *not an upper bound of* $\{a_i, a_j\}$ which could potentially dominate a_m and destroy the required equality

$$a_1 = h_m(x) = h_m(y) \sqcap g_{i,j}(x) = a_m \sqcap g_{i,j}(x).$$

By inspecting the explicit definition of $g_{i,j}$ given immediately before this lemma, we see that for $g_{i,j}(x)$ to assume a value other than a_i, a_j, or a_1, x must be above a mub of $\{a_i, a_j\}$. But since $x \sqsubseteq y$ and $y \sqsupseteq a_0$, x and a_0 are compatible upper bounds of $\{a_i, a_j\}$. On the other hand, D is a meet cpo and so distinct mub's of $\{a_i, a_j\}$ must be incompatible, by Lemma 2.1. This implies $x \sqsupseteq a_0$, contradicting the assumption that x comes from region β.

Next we show that $g_{i,j}$ is a *minimal* upper bound of $\{h_k, h_l\}$. Suppose g is a stable function such that

$$h_k \sqsubseteq_s g \sqsubseteq_s g_{i,j} \quad \text{and} \quad h_l \sqsubseteq_s g \sqsubseteq_s g_{i,j}.$$

We show that $g = g_{i,j}$, and hence $g_{i,j}$ is minimal among the upper bounds of $\{h_k, h_l\}$.

The assumptions in the previous paragraph imply in particular that

$$h_k(a_0) \sqsubseteq g(a_0) \sqsubseteq g_{i,j}(a_0).$$

Hence $a_k \sqsubseteq g(a_0) \sqsubseteq a_0$. Similarly, $a_l \sqsubseteq g(a_0) \sqsubseteq a_0$. This means that $g(a_0)$ is an upper bound of $\{a_k, a_l\}$, compatible with a_0. But a_0 is already a minimal upper bound of $\{a_k, a_l\}$. We must have $g(a_0) = a_0$, again by Lemma 2.1.

By Item (a) in Lemma 2.2, keeping in mind that g and $g_{i,j}$ are compatible, we have $g(x) = g_{i,j}(x)$ for all $x \sqsubseteq a_0$. In particular, $g(a_1) = a_1$, $g(a_i) = a_i$, and $g(a_j) = a_j$. From the two latter equalities, it follows that $g(z) = g_{i,j}(z)$ for any minimal upper bound z of $\{a_i, a_j\}$. Therefore, $g(x) = g_{i,j}(x)$ for all elements in the $(\mathsf{mub}, \mathsf{meet})$-closure of $\{a_i, a_j\}$. By Item (b) of Lemma 3.2, we have $g = g_{i,j}$. □

Theorem 4.1 *Suppose D is an ω-algebraic meet-cpo with property* M *which satisfies the finite descending chain and finite ascending chain conditions, but fails the finite antichain condition. Then the stable function space $[D \to D]$ fails* M*, i.e., there exists a finite set of compact stable functions with an infinite number of minimal upper bounds.*

Proof. By Lemma 4.5, the set $\{h_2, h_3\}$ of compact elements in $[D \to D]$ has infinitely many minimal upper bounds $g_{i,j}$ for all distinct $i, j \geq 4$. □

5 The finite (mub, down)-closure condition

What is left to be done for a complete solution of the problem is the second step, mentioned earlier: to show that if D violates axiom MI^∞ , then the function space $[D \to D]$ violates axiom M or I. We have not obtained a solution for the second step, but many examples lead us to conjecture that the answer should be on the affirmative. We highlight two such examples in this section, both can be ruled as "forbidden structures" by Lemma 5.1.

Property MI^∞ states that a finite set of compact elements has a finite (mub, down)-closure. The dark nodes in Fig. 4 and Fig. 5 form finite subsets of compact elements whose (mub, down)-closures are infinite. Therefore, both domains in Fig. 4 and Fig. 5 violate MI^∞ . Fig. 4 grows infinitely tall, while Fig. 5 grows infinitely wide (the latter can be found in [1]).

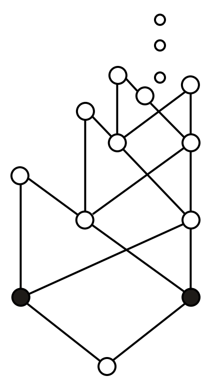

Fig. 4. Infinitely tall (mub, down)-closure

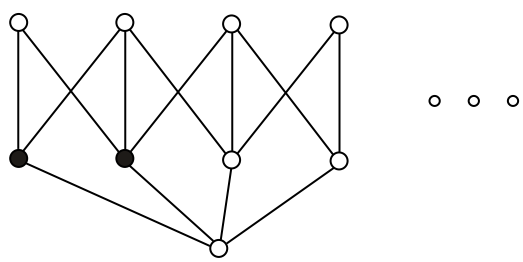

Fig. 5. Infinitely wide (mub, down)-closure

The next lemma provides a rather general method to show why the domains in Fig. 4 and Fig. 5 are forbidden.

Lemma 5.1 *Let D be an ω-algebraic meet-cpo with property* M. *Suppose*

(1) the (mub, meet)*-closure of every finite down-closed subset of D is finite,*
(2) the (mub, down)*-closure of B_0 is infinite for some finite set B_0 of compact elements of D, and for each element $b \in B_0$, the* (mub, down)*-closure of b,* $(\mathsf{mub}, \mathsf{down})^*\{b\}$, *is finite.*

Then $[D \to D]$ *violates axiom* M.

One can check that both Fig. 4 and Fig. 5 satisfy the two conditions in this lemma, with B_0 consisting of the dark nodes in the pictures, respectively. They are therefore both forbidden structures.

Proof. Suppose B_0 is a finite set of compact elements with the properties given in item (2) of the lemma. Consider the following sets for all $i \geq 1$:

$$A_i := (\mathsf{mub}, \mathsf{down})(B_{i-1}),$$
$$B_i := (\mathsf{mub}, \mathsf{meet})^*(A_i).$$

All A_i, B_i are finite (sets of compact elements): each A_i is finite because it is obtained by *one application* of the $(\mathsf{mub}, \mathsf{down})$-operation on B_{i-1} (and *not the* $(\mathsf{mub}, \mathsf{down})$-*closure* of B_{i-1} which may become infinite); each B_i is finite because of condition (1) of the lemma. Moreover, B_i is a *proper subset* of B_{i+1} for each $i \geq 1$ because $(\mathsf{mub}, \mathsf{down})^*(B_0)$ is infinite (if $B_i = B_{i+1}$ for some $i \geq 1$, then B_i is a finite $(\mathsf{mub}, \mathsf{down})$-closed set already).

We show for sufficiently large k and each $i \geq k$, ϕ_{B_i} is a minimal upper bound of the set $F(B_0)$ of compact stable functions determined by the $(\mathsf{mub}, \mathsf{down})$-closure of elements in B_0, where

$$F(B_0) := \{\phi_{(\mathsf{mub}, \mathsf{down})^*\{b\}} \mid b \in B_0\}.$$

For sufficiently large k, we have $((\mathsf{mub}, \mathsf{down})^*\{b\}) \cap B_k = (\mathsf{mub}, \mathsf{down})^*\{b\}$ for each $b \in B_0$ because B_0 is finite and each $(\mathsf{mub}, \mathsf{down})^*\{b\}$ is finite for $b \in B_0$, by condition (2). It follows from Lemma 3.3 that for all $i \geq k$, ϕ_{B_i} is an upper bound of $F(B_0)$. To show that such a ϕ_{B_i} is minimal (upper bound for $F(B_0)$), suppose for some ϕ we have $g \sqsubseteq_s \phi \sqsubseteq_s \phi_{B_i}$ for all $g \in F(B_0)$. By the definition of B_i, for each $b \in B_0$, there exists a $g \in F(B_0)$ such that $g(b) = b$ and $\phi_{B_i}(b) = b$ by the definition of B_i (note that $i \geq k$). Thus $\phi(b) = \phi_{B_i}(b)$ for all $b \in B_0$. Taking this as the induction basis, one can show (using the same technique as in the proof of Lemma 3.4) by induction on j that $\phi(a) = \phi_{B_i}(a)$ for all $a \in B_j$ with $j \geq 0$. Therefore, $\phi = \phi_{B_i}$ by Lemma 3.2. Since all B_i, and hence all ϕ_{B_i} with $i \geq k$ are distinct from each other, $[D \to D]$ violates M. □

Note the technique described in the above lemma do not directly apply to our earlier example in Fig. 2, because condition (1) of the lemma is not satisfied. However, the infinite chain in the middle of that figure determines an infinite chain of compact stable functions below the one generated by the three beginning elements (marked by dark nodes), in light of Lemma 3.1, Lemma 3.3, and

Lemma 3.4. Therefore, Fig. 2 is also a forbidden structure: its stable function function space violates I.

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